SURVEY OF NUMERICAL METHODS FOR SOLVING TIME-VARYING FUSE EQUATIONS

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Principal Nomenclature

c	Specific heat
ρ	Density
T_A	Ambient temperature
T	Fuse element temperature
T_m	Fuse element melting temperature
σ	Electrical conductivity
σ_A	Ambient value of electrical conductivity
k	Thermal conductivity
K	Thermal difusivity $(k/c\rho)$
Δt	Time step length
t	Time
Δx	Step length
I	Prospective current
a	Cross-sectional area
J	Current density
α	Temperature coefficient of Resistance
i, n	Generalised space and time identifiers (T_i^n corresponds to temperature at point $i\,\Delta x$ along element at time $n\,\Delta t$ seconds)

1. INTRODUCTION Computer prediction of fuse characteristics have obvious labour saving benefits and advantages in computing fuse performances which are difficult or impossible to determine from tests.

Prior to digital computers equations of the form of (1) for simulating prearcing performances of fuses were impossible to solve accurately for all but the very simplest of fuse geometries unless gross reductions were made in fuse representations. The advent of high speed digital computers and powerful numerical methods has greatly enhanced equation solving capability enabling improved predictions of fuse performance.

The ideal pre-requisites of numerical methods for solving equations by computer are:

- High Accuracy
- Guaranteed Numerical Stability
- Low Computer Running Times
- Minimum Complexity
- Low Computer Storage

This paper presents results of a study of numerical methods for solving

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parabolic equations to enable selection of the most suitable numerical methods for solving the pre-arcing performance of fuses subject to the stated idealised numerical constraints.

$$c\rho \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\frac{(k\partial T)}{\partial x} + \frac{\partial}{\partial y}\frac{(k\partial T)}{\partial y} + \frac{\partial}{\partial z}\frac{(k\partial T)}{\partial z} + \frac{J^2}{\sigma} \qquad \ldots (1)$$

2. <u>NUMERICAL METHODS STUDIED</u> Fuse numerical solutions involve dividing fuse elements into small sections and determining the temperature of each section over discrete time intervals. By this technique the partial differential equations governing joulean heat flow in fuse elements may be approximated by difference equations using surrounding temperature values and temperature rise calculated in terms of space and time ordinates for all fuse geometries. This treatment reduces (1) to sets of simultaneous algebraic equations which may be solved at successive time intervals using a numerical method.

Three broad classes of numerical method exist for solving the type of algebraic equations resulting from fuse modelling. These methods are:

Matrix methods
Finite Element methods
Finite Difference methods

Upon examination Matrix and Finite Element methods were found to require much greater storage than the Finite Difference variety as numerical coefficients in the latter method's algorithms are implicit and thus do not need storing. In addition Finite Difference methods were found to be more flexible for generalised solutions and simpler to apply.

The Finite Difference methods were consequently preferred for solving fuse equations. These methods were applied to time-varying fuse equations and critically examined against the criteria established in Section 1 by comparing computed results with analytical solutions of equation (1).

A simple example of electro-thermal heat flow was used for comparison purposes. The example was that of a thermally insulated uniform section current carrying conductor with ends held at zero temperature. Time-varying solutions are feasible for this problem providing that heat is generated at a constant rate[1]. The heat flow for this case is one dimensional and therefore governed by (2).

$$c\rho \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \frac{J^2}{\sigma} \qquad \ldots (2)$$

The analytical solution for temperature along the conductor is given by (3). The general form of temperature variation with time and position along the conductor is shown in figure 1 for general values of time equal to (n-1) Δt, n Δt and (n+1) Δt.

$$T = \frac{J^2 \ell^2}{2k\sigma}\left\{1 - \left(\frac{x}{\ell}\right)^2 - \frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3}\cos\frac{(2n+1)\pi x}{2\ell}\, e^{-\frac{\pi k t(2n+1)}{4c\rho\ell}}\right\} \qquad \ldots (3)$$

Temperature may be monitored along the conductor at discrete points Δx at time intervals Δt, 2 Δt ... (n-1)Δt, nΔt, etc., as indicated. Numerical solution of the same problem requires replacement of derivatives $\partial T/\partial t$ and $\partial^2 T/\partial x^2$ in (2) by difference approximations. For instance $\partial^2 T/\partial x^2$ may be deduced at all points along the conductor at time $n\Delta t$ in terms of temperature at generalised points $(i-1)\Delta x$, $i\Delta x$ and $(i+1)\Delta x$ giving

$$\left(\frac{\partial^2 Ti}{\partial x^2}\right)^n = \frac{(T^n_{i+1} + T^n_{i-1} - 2T^n_i)}{\Delta x^2} \qquad \ldots (4)$$

The time derivative at all points $i\Delta x$ at time $n\Delta t$ consequently becomes

$$\left(\frac{\partial Ti}{\partial t}\right)^n = \frac{K}{cp}(T^n_{i+1} + T^n_{i-1} - 2T^n_i) + \frac{J^2}{cp\sigma} \qquad \ldots (5)$$

This approximation is made at all points along the conductor and leads to $(\ell/\Delta x - 1)$ simultaneous equations for specifying temperature along the conductor.

The time derivative may be approximated also by differences, the simplest of which is

$$\left(\frac{\partial Ti}{\partial t}\right)^n = \frac{(T^{n+1}_i - T^n_i)}{\Delta t}$$

and leads to the Euler formulation and Explicit numerical prediction method for T_i at $t = (n+1)\Delta t$

i.e. $$T^{n+1}_i = T^n_i + \left(\frac{\partial T_i}{\partial t}\right)^n \Delta t \qquad \ldots \text{ Euler Method}$$

from which

$$T^{n+1}_i = T^n_i + M(T^n_{i+1} + T^n_{i-1} - 2T^n_i) + \frac{J^2\,\Delta t}{cp\sigma}$$

$M = \dfrac{k\,\Delta t}{cp\,\Delta x^2}$ and is termed the modal parameter.

The Explicit method predicts T' for T^{n+1} in figure 2.

An alternative formulation uses forward difference formula and results in the Implicit (Laasonen) method.

$$T^{n+1}_i = T^n_i + \left(\frac{\partial T_i}{\partial t}\right)^{n+1} \Delta t$$

$$T^{n+1}_1 = T^n_i + M(T^{n+1}_{i-1} + T^{n-1}_{i-1} - 2T^n_i) + \frac{J^2\,\Delta t}{cp\,\sigma} \qquad \ldots \text{ Implicit Method}$$

In this case T^{n+1}_i is solved iteratively at each time step at all points along the conductor. The Implicit method predicts T" for T^{n+1}_i in figure 2.

Clearly from inspection of T^{n+1}_i predictions with the Explicit and Implicit methods a more accurate formulation would be the average of both predictions. This formulation leads to the 2nd order Runge Kutta formula and Crank Nicholson method.

$$T_i^{n+1} = T_i^n + \tfrac{1}{2}\left\{ \left(\frac{\partial T}{\partial t}\,i\right)^{n+1} + \left(\frac{\partial T}{\partial t}\,i\right)^n \right\} \Delta t$$

giving

$$T_i^{n+1} = T_i^n + \frac{M}{2}\left\{ T_{i+1}^{n+1} + T_{i-1}^{n+1} + T_{i+1}^{n} + T_{i-1}^{n} - 2(T_i^{n+1} + T_i^n) \right\} + \frac{J^2\,\Delta t}{cp\,\sigma}$$

The Crank Nicholson algorithm is implicit in T_i^{n+1} and T_{i+1}^{n+1} and must also be solved iteratively. The Crank Nicholson method predicts T''' for T_i^{n+1} in figure 2.

A variation in the Implicit method was used by Leach, Newbery and Wright[2].

Here $T_i^{n+\frac{1}{2}}$ is computed at $t = (n+\frac{1}{2})\,\Delta t$ using the Standard Implicit method and T_i^{n+1} obtained by linear extrapolation using $T_i^{n+\frac{1}{2}} = \frac{1}{2}(T_i^{n+1} + T_i^n)$.

Another method used for solving parabolic equations is the "Du Fort Frankel" method. This method uses central difference formulae over the interval 2 Δt viz:

$$\left(\frac{\partial T_i}{\partial t}\right)^n = \frac{(T_i^{n+1} - T_i^{n-1})}{2\,\Delta t} \quad \text{and} \quad T_i^n = \frac{(T_i^{n+1} + T_i^{n-1})}{2}$$

whereupon

$$T_i^{n+1} = T_i^{n-1} + 2M(T_{i+1}^{n+1} + T_{i-1}^{n+1} - T_i^{n+1} - T_i^{n-1}) + \frac{2J^2\,\Delta t}{cp\,\sigma}$$

3. <u>BASIS FOR COMPARISON</u> The above methods were compared with analytical solutions for temperature along uniform current carrying conductors as specified in section 2. The solution for time-varying temperature distribution is given by (3) and shown in figure 3. Numerical solutions were obtained for each of the methods discussed and the Crank Nicholson and Leach et al. methods were found to be identical and the most accurate for this problem. Errors increased as prediction time lengthened and as the step length Δx increased for all methods, figure 4.

Prediction of prospective current versus melting time was more accurate, figure 5, which is to be expected from inspection of respective expressions for maximum temperature along the conductor and melting current.

$$\hat{T} = \frac{I^2\,\ell^2}{2a^2 K\sigma}\left\{ 1 - \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^3}\, e^{-\frac{k(2j+1)^2\pi^2 t}{4cp\ell^2}} \right\}$$

$$\frac{I_{MFC}}{I} = \left\{ 1 - \frac{32}{\pi^3} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^3}\, e^{-\frac{k(2n+1)^2\pi^2 t}{4cp\ell^2}} \right\}^{\frac{1}{2}}$$

where $I_{MFC} = \left(\frac{2KT_m a^2 \sigma}{\ell^2}\right)^{\frac{1}{2}}$

4. ACCURACY OF METHODS Errors in numerical solutions occur due to 'round-off' and 'truncation' errors. Round-off errors are produced upon rounding off computed values to fixed decimal places and are not normally problematical with modern computers. Greatest error is introduced by approximating derivatives by simple forward and backward difference formulae and is termed truncation error. The error occurs at each computation step and may propagate to excessive proportions unless controlled.

Truncation error is assessible upon expanding T_i^{n+1} in Taylors series in both time and space ordinates. For example in the one dimensional Explicit formulation considered in section 2 the truncation error is

$$= \left\{\frac{\partial T}{\partial t} - \frac{K}{cp}\frac{\partial^2 T}{\partial x^2} - \frac{J^2}{cp\sigma}\right\} - \left\{\frac{(T_i^{n+1} - T_i^n)}{\Delta t} - \frac{K}{cp}(T_{i+1}^n + T_{i-1}^n - 2T_i^n) + \frac{J^2}{cp\sigma}\right\}$$

$$= \tfrac{1}{2}\Delta t\left(\frac{\partial^2 T_i}{\partial t^2}\right)^n + \frac{K}{cp}\frac{\Delta x^2}{12}\left(\frac{\partial^4 T_i}{\partial x^4}\right)^n$$

The coefficients of Δt and Δx^2 in the error term are bounded because of the continuity of the partial derivatives and will take values depending upon the problem under investigation. The truncation error $= K_1\Delta t + K_2(\Delta x)^2$ for all sufficiently small Δt and Δx. Usual practise expresses the truncation error and implied conditions symbolically as

$$O[\Delta t] + O[(\Delta x)^2]$$

The truncation errors for the other numerical methods are determined in similar manner and are as summerised.

	Truncation error
Implicit Method	$O[\Delta t] + O[(\Delta x)^2]$
Crank Nicholson Method	$O[(\Delta t)^2] + O[(\Delta x)^2]$
Du Fort Frankel Method	$O[(\Delta t)^2] + O[(\Delta x)^2] + O[(\frac{\Delta t}{\Delta x})^2]$

Clearly for good accuracy Δx and Δt should be arranged to be much less than unity and for the Du Fort-Frankel method $\Delta t << \Delta x$.

The errors were found consistent with computed results where the Crank Nicholson method was the most accurate and the Du Fort Frankel the least. The poor accuracy of the Du Fort Frankel method was expected as the method requires a starting method to obtain T_i^{n-1} at the first time step and $(\Delta t/\Delta x)$ was finite.

5. BEHAVIOUR OF NUMERICAL METHODS In solving solutions over long times it is desirable to use the largest possible time step to minimise computer running time. Increase of Δt for a given fuse model however increases the modal parameter M and affects the accuracy and behaviour of solution. Moreover if M is increased indiscriminately solutions may oscillate giving false convergence and in some cases become instable.

The numerical behaviour of all the methods were studied using Richtmeyers[4] generalised stability analysis and test runs with each method. Richtmeyers stability analysis covers numerical formulations of the form

$$T_i^{n+1} = T_i^n + \left\{ \theta \left(\frac{\partial T_i}{\partial t} \right)^{n+1} + (1-\theta) \left(\frac{\partial T_i}{\partial t} \right)^n \right\} \Delta t$$

The formulation is suitable for investigating the behaviour of the Explicit, Implicit and Crank Nicholson methods as $\theta = 0, 1, \frac{1}{2}$ corresponds to each method respectively.

Richtmeyers Analysis though general is valid only for equations of the form

$$\frac{\partial T}{\partial t} = \frac{K}{cp} \frac{\partial^2 T}{\partial x^2} \quad \ldots (6)$$

The equation though simpler than (2) for one dimensional joulean heat flow in conductors is useful in providing guidance on stability performance of the methods considered, however for practical fuse modelling numerical behaviour should also be assessed from test runs with various time steps.

Analytical solution of (6) is

$$T_i^n = \sum_{m=-\infty}^{\infty} A_{(m)} \, e^{jmi\,\Delta x} \left(\xi_{(m)} \right)^n \quad \ldots (7)$$

where
$$A_{(m)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \psi(x) \, e^{-jmx} \, dx$$

and $\xi_{(m)}$ is termed the growth factor and is given as

$$\xi_{(m)} = \left\{ \frac{1 - 2M(1-\theta)\left[1 - \cos(m\,\Delta x)\right]}{1 + 2M\theta \left[1 - \cos(m\,\Delta x)\right]} \right\}$$

From inspection of (7) T_i^n is convergent providing $\xi_{(m)} < \pm 1$. $\xi_{(m)}$ however varies with θ and therefore with numerical method.

Behaviour of $\xi_{(m)}$ is shown for the Explicit, Implicit and Crank Nicholson methods, figure 6 and the following important constraints apply.

Numerical Method	Limiting Value of M	
	Smooth Convergence	Stable Oscillation
Explicit	.25	0.5
Implicit	∞	-
Crank Nicholson	1	∞
Du Fort Frankel	-	∞

Clearly all methods except the Explicit method are unconditionally stable.

The methods were examined by increasing Δt until oscillations occurred using a practical fuse model. In the case of the Crank Nicholson method small oscillations were observed in solutions when Δt was increased to excessive step lengths of the order of 10s. All the other methods behaved as predicted by Richtmeyers analysis.

Rate of convergence is important in minimising computer running times. Again according to Richtmeryer[4] convergence rate is related to solution trunction error and generally the lower the error the faster the convergence. From figure 6 it is clear that the Implicit method though always smoothly convergent has larger errors than the Crank Nicholson method and is therefore relatively slowly convergent.

6. DISCUSSION Several numerical methods have been presented for solving fuse electro-thermal equations and were assessed against analytical solutions for comparison purposes. Although the chosen analytical solutions bear little resemblance to actual fuses, they do permit establishment of some general guide lines on the numerical behaviour of solutions of fuse equations.

Other numerical methods termed 'Multi Time Step' methods[4] were also considered. These methods were theoretically slightly more accurate than the methods presented in this paper but storage was at least twice the maximum storage for the presented methods and more complex to program, as 'Multi Time Step' varieties involve storing and computing with temperature values at three or more times steps at each time interval.

The method found most suitable from these and subsequent studies was the Crank Nicholson method even though the Explicit method was superior in accuracy, convergence and storage. The Explicit method was rejected for calculating conductor temperature as the method became unstable at low values of M which limited applications to exceptionally small values of Δt.

The methods were checked for the more practical fuse case where electrical conductivity varied with temperature. The problem investigated was identical to that specified in section 2 except that

$$\sigma(T) = \frac{\sigma_A}{(1+\alpha(T_i^n - T_A))}$$

The results using the presented methods were again compared and two interesting findings made.

(a) The method used by Leach et al[3], which gave identical results with the Crank Nicholson method when joulean heat generated was constant, gave lower temperature predictions than the Explicit, Implicit, Crank Nicholson and Du Fort Frankel methods. This result may be expected as the method used by Leach et al. assumes temperature rises identically over both halves of the time interval. The Du Fort Frankel method was again the least accurate.

(b) Predictions of times to reach melting temperature using σ_{AV} established at $(T_m+T_A)/2$ and $\sigma(T)$ were in reasonable agreement for short melting times but differed increasingly as melting times increased. This finding is of interest as it demonstrates that for the conditions stated the 'mean electrical conductivity' value may be used in some short-circuit calculations without great error, (figure 7).

7. CONCLUSIONS Numerical methods for predicting fuse characteristics have been assessed against the desirable criteria of high accuracy, stability, low computer storage and running times with minimum complexity. The method of Finite Differences was found superior to other methods and the Crank Nicholson formulation the most suitable for solving fuse equations.

8. REFERENCES

1. Carslaw, H.S., Jeager, J.C. : "Conduction of Heat in Solids', Pub. Clarendon Press, Oxford, 1959.

2. Leach, J.G., Newbery, P.G., Wright, A. ; "Analysis of HRC Fuselink Prearcing Performance by a Finite Difference Method", Proc. IEE, Vol. 120, No.9, September 1973.

3. Richtmeyer, R., Morgan, K.W. : "Difference Methods for Initial Value Problems", Pub. Interscience, 2nd Edition.

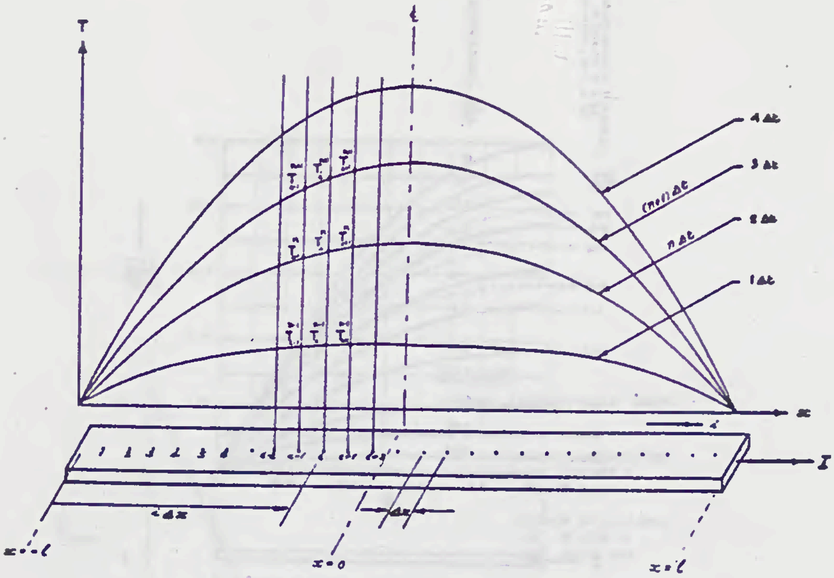

FIGURE 1 TEMPERATURE : TIME DISTRIBUTION ALONG THERMALLY INSULATED CURRENT CARRYING CONDUCTOR

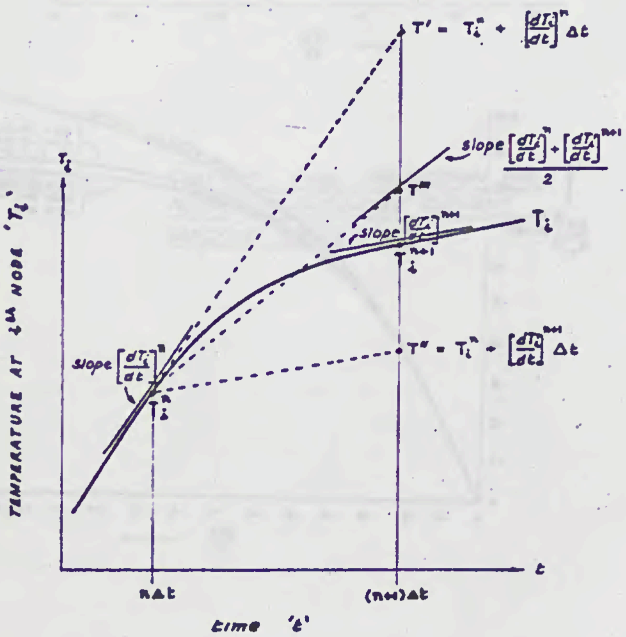

FIGURE 2 GRAPHICAL DEMONSTRATION OF NUMERICAL PREDICTION METHODS

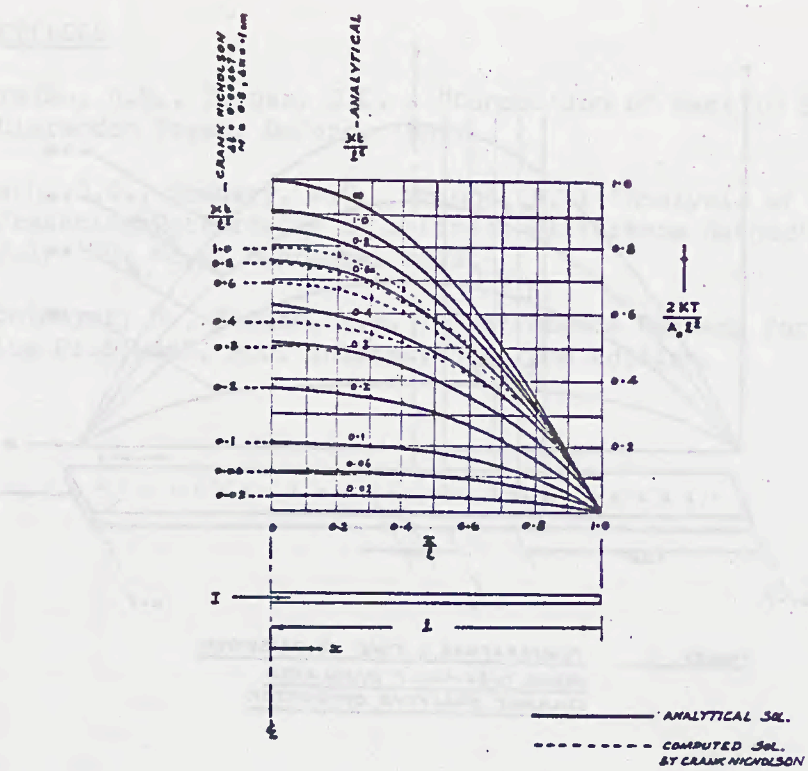

FIGURE 3 COMPUTED AND ANALYTICAL SOLUTIONS FOR THERMALLY INSULATED CONDUCTORS

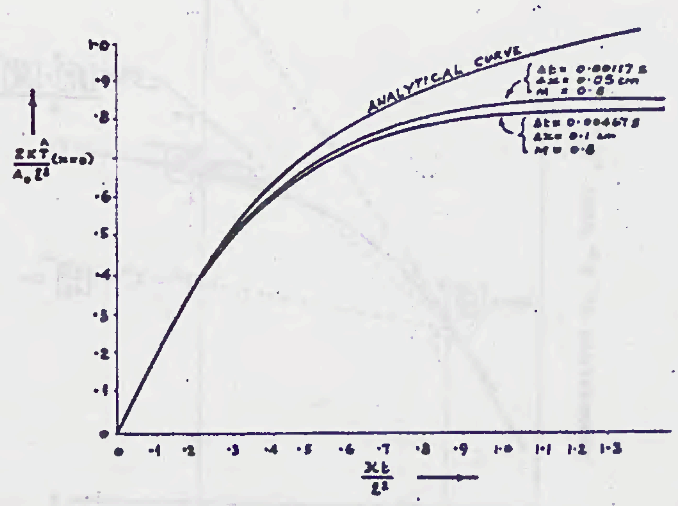

FIGURE 4 COMPARISON OF ANALYTICAL AND CRANK NICHOLSON SOLUTIONS FOR THERMALLY INSULATED CURRENT CARRYING CONDUCTOR PROBLEM

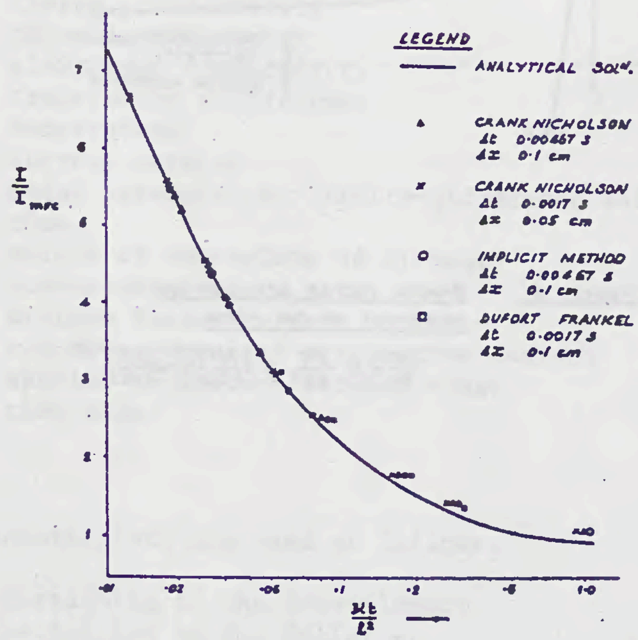

FIGURE 5 CURRENT : TIME PREDICTION FOR THERMALLY INSULATED CURRENT CARRYING CONDUCTORS

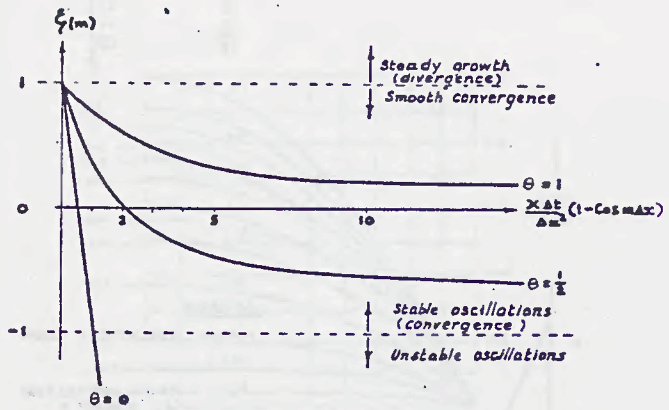

FIGURE 6 GROWTH FACTOR BEHAVIOR FOR EQUATIONS OF THE FORM

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \doteq \theta\left[\frac{\partial T}{\partial t}\right]^{n+1} + (1-\theta)\left[\frac{\partial T}{\partial t}\right]^{n-1}$$

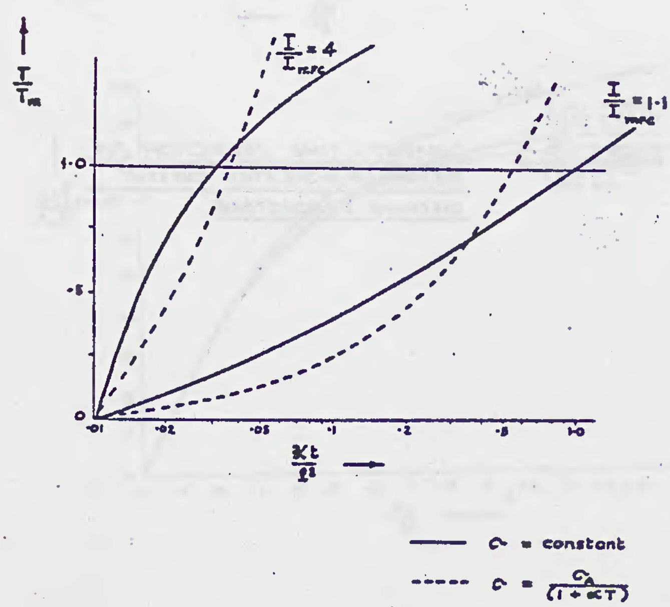

FIGURE 7 TEMPERATURE PREDICTION PROFILES FOR CONSTANT AND TEMPERATURE VARYING ELECTRICAL CONDUCTIVITY